STAT 429 Time Series Problem Set I Solution

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Problem 1.2 (a) Solution:

$$\begin{aligned}
&E[(X_{n+1} - f(X_1, \cdots, X_n))^2 | X_1, \cdots, X_n] \\
=&E[(X_{n+1} - E(X_{n+1}) + E(X_{n+1}) - f(X_1, \cdots, X_n))^2 | X_1, \cdots, X_n] \\
=&Var(X_{n+1} | X_1, \cdots, X_n) + (E(X_{n+1} | X_1, \cdots, X_n) - f(X_1, \cdots, X_n))^2
\end{aligned} \tag{1}$$

Therefore, minimize the function $E[X_{n+1} - f(X_1, \cdots, X_n)^2 | X_1, \cdots, X_n]$ is equivalent to minimize $((E(X_{n+1}) - f(X_1, \cdots, X_n))^2 | X_1, \cdots, X_n)$

Then we have $f(X_1, \cdots, X_n) = E(X_{n+1} | X_1, \cdots, X_n)$

Problem 1.2 (b) Solution:

$$\begin{aligned}
&\min_f E[(X_{n+1} - f(X_1, \cdots, X_{n+1}))^2] \\
=&\min_f E[E[(X_{n+1} - f(X_1, \cdots, X_n))^2] | X_1, \cdots, X_n] \\
=&\min_f E[Var(X_{n+1} | X_1, \cdots, X_n|) + (E(X_{n+1}) - f(X_1, \cdots, X_n))^2 | X_1, \cdots, X_n] \\
=&E[Var(X_{n+1} | X_1, \cdots, X_n|)] + \min_f E[(E(X_{n+1}) - f(X_1, \cdots, X_n))^2 | X_1, \cdots, X_n]
\end{aligned} \tag{2}$$

And because $(E(X_{n+1}) - f(X_1, \cdots, X_n))^2 \geq 0$, that means $E[(E(X_{n+1}) - f(X_1, \cdots, X_n))^2 | X_1, \cdots, X_n] \geq 0$, therefore, the minimal value should be 0, when $f(x_1, \cdots, X_n) = E[X_{n+1} | X_1, \cdots, X_n]$

Problem 1.2 (c) Solution: Directly use Problem 1.2 (b), we get the answer should be $E[X_{n+1} | X_1, \cdots, X_n] = \mu$. for $X_1, X_2, \cdots$ are IID with mean 0 and variance σ^2

Problem 1.2 (d) Solution: Suppose the estimator is $\hat{\mu} = \sum_{i=1}^n d_i X_i$, where d_is are constant numbers.Therefore, by unbiased condition, we have

$$E(\hat{\mu}) = \sum_{i=1}^n d_i E(X_i) = \mu \sum_{i=1}^n d_i \tag{3}$$

We can get $\sum_{i=1}^{n} d_i = 1$ In order to have best unbiased estimator, we need the smallest variance. And

$$Var(\hat{\mu}) = Var(\sum_{i=1}^{n} d_i X_i) = \sigma^2 \sum_{i=1}^{n} d_i^2 \tag{4}$$

Therefore

$$\min(Var(\hat{\mu})) \equiv \min \sum_{i=1}^{n} d_i^2 \tag{5}$$

Condition on $\sum_{i=1}^{n} d_i = 1$

$$\min \sum_{i=1}^{n} d_i^2 \equiv \min \sum_{i=1}^{n} |d_i|^2 \tag{6}$$

And

$$\sum_{i=1}^{n} |d_i|^2 \geq (\sum_{i=1}^{n} |d_i|)^2 * \frac{1}{n} \geq (\sum_{i=1}^{n} d_i)^2 * \frac{1}{n} = \frac{1}{n} \tag{7}$$

The equality holds when all the d_is are $\frac{1}{n}$, that means our unbiased estimator is $\overline{X}$

Problem 1.2 (e) Solution: Similar as Problem 1.2 (d). Here we need suppose our predictor $\tilde{X}_{n+1} = \sum_{i=1}^{n} d_i X_i$, with $\sum_{i=1}^{n} d_i = 1$, in order to be unbiased for μ.

Then

$$\begin{aligned}
&E(X_{n+1} - \sum_{i=1}^{n} d_i X_i) \\
=&E(X_{n+1} - \overline{X} + \overline{X} - \sum_{i=1}^{n} d_i X_i) \\
=&E(X_{n+1} - \overline{X})^2 + 2E(X_{n+1} - \overline{X})(\overline{X} - \sum_{i=1}^{n} d_i X_i) + E(\overline{X} - \sum_{i=1}^{n} d_i X_i)^2
\end{aligned} \tag{8}$$

The second term:

$$
\begin{aligned}
&E(X_{n+1}-\overline{X})(\overline{X}-\sum_{i=1}^{n}d_iX_i)\\
=&Cov(X_{n+1}-\overline{X},\overline{X}-\sum_{i=1}^{n}d_iX_i)\\
=&Cov(X_{n+1},\overline{X})+Cov(\overline{X},\sum_{i=1}^{n}d_iX_i)-Cov(X_{n+1},\sum_{i=1}^{n}d_iX_i)-Cov(\overline{X},\overline{X})\\
=&\cdots\\
=&\frac{\sigma^2}{n}\sum_{i=1}^{n}d_i-\sum_{j=1}^{n}(\frac{1}{n})^2\sigma^2\\
=&0
\end{aligned}
\tag{9}
$$

Then, we get

$$
E(X_{n+1}-\sum_{i=1}^{n}d_iX_i)^2\geq E(X_{n+1}-\overline{X})^2 \tag{10}
$$

for any $\sum_{i=1}^{n}d_iX_i$, i.e. $E(X_{n+1}-\sum_{i=1}^{n}d_iX_i)^2$ minimize at $\sum_{i=1}^{n}d_iX_i=\overline{X}$

Problem 1.2 (f) Solution: From Problem 1.2 (a), the minimum MSE predictor of S_{n+1} given $S_1,\cdots,S_n$ is:

$$
\begin{aligned}
E(S_{n+1}|S_1,\cdots,S_n)=&E(S_n+X_{n+1}|S_1,\cdots,S_n)\\
=&S_n+E(X_{n+1}|S_1,\cdots,S_n)\\
=&S_n+E(X_{n+1})\\
=&S_n+\mu
\end{aligned}
\tag{11}
$$

Problem 1.4 solution

(a)Yes, it is a stationary process. $\mu_X=a$

$$
\gamma_X(h)=Cov(X_h,X_0)=\begin{cases}(b^2+c^2)\sigma^2, & \text{if } h=0;\\ bc\sigma^2, & \text{if } |h|=2;\\ 0 & \text{otherwise.}\end{cases}
$$

(b)Yes, it is a stationary process. $\mu_X=0$ and $\gamma_X(h)=\cos(ch)\sigma^2$

(c) If $C\neq k\pi$ NO, it is not a stationary process. $\mu_X=0$

$$
\begin{aligned}
\gamma_X(t+h,t)=&E[Z_t\cos(ct)+Z_{t-1}\sin(ct)][Z_{t+h}\cos(c(t+h))+Z_{t+h-1}\sin(c(t+h))]\\
=&\cos(ct)\cos(c(t+h))E[Z_tZ_{t+h}]+\cos(ct)\sin(c(t+h))E[Z_tZ_{t+h-1}]\\
&+\cos(c(t+h))\sin(ct)E[Z_{t-1}Z_{t+h}]+\sin(c(t+h))\sin(ct)E[Z_{t-1}Z_{t+h-1}]
\end{aligned}
\tag{12}
$$

Therefore, we have:

$$\gamma_X(t+h,t) = Cov(X_{t+h}, X_t) = \begin{cases} \sigma^2 & \text{if } h = 0; \\ \cos(ct)\sin(c(t+1))\sigma^2 & \text{if } h = 1; \\ \cos(c(t-1))\sin(ct)\sigma^2 & \text{if } h = -1; \\ 0 & \text{otherwise} \end{cases}$$

(d) Yes, it is stationary, $\mu_X(t) = a$, and $\gamma_X(t+h,t) = b^2\sigma^2$

(e) If $c \neq k\pi$,then it is not stationary. $\mu_X(t) = 0$ and

$$\begin{aligned} \gamma_X(t+h,t) =& E[Z_0^2]\cos(c(t+h))\cos(ct) \\ =& \cos(ct+ch)\cos(ct)\sigma^2 \end{aligned} \tag{13}$$

(f) Yes, it is stationary. And $\mu_X(t) = 0$

$$\gamma_X(h) = Cov(X_{h+t}, X_t) = \begin{cases} \sigma^4, & \text{if } h = 0; \\ 0, & \text{if } |h| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Problem 1.5 Solution

(a) We can directly use our result in Problem 1.4 (a) with $a = 0, b = 1, c = \theta = 0.8$. Therefore, the answer is $\mu_X(t) = 0$ and

$$\gamma_X(h) = Cov(X_{h+t}, X_t) = \begin{cases} 1.64, & \text{if } h = 0; \\ 0.8 & \text{if } |h| = 2; \\ 0 & \text{otherwise.} \end{cases}$$

And the correspondence autocorrelation func. is:

$$\rho_X(h) = Cor(X_{h+t}, X_t) = \begin{cases} 1, & \text{if } h = 0; \\ \frac{0.8}{1.64} & \text{if } |h| = 2; \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$\begin{aligned} & Var[\frac{1}{4}(X_1 + X_2 + X_3 + X_4)] \\ =& \frac{1}{16}E(X_1 + X_2 + X_3 + X_4)^2 \\ =& \frac{1}{16}E(0.8Z_0 + 1.8Z_1 + 0.8Z_{-1} + 1.8Z_2 + Z_3 + Z_4)^2 \\ =& \frac{1}{16}(0.8^2 + 1.8^2 + 0.8^2 + 1.8^2 + 1 + 1) \\ =& 0.61 \end{aligned} \tag{14}$$

(c) When $\theta = -0.8$

$$
\begin{aligned}
&Var[\frac{1}{4}(X_1 + X_2 + X_3 + X_4)] \\
=&\frac{1}{16}E(X_1 + X_2 + X_3 + X_4)^2 \\
=&\frac{1}{16}E(-0.8Z_0 + 0.2Z_1 - 0.8Z_{-1} + 0.2Z_2 + Z_3 + Z_4)^2 \\
=&\frac{1}{16}(0.8^2 + 1.8^2 + 0.2^2 + 0.2^2 + 1 + 1) \\
=&0.21
\end{aligned}
\tag{15}
$$

Problem 1.14 Solution:

To show the filter passes third-degree polynomials;

$$
\begin{aligned}
\sum(a_j) &= \frac{1}{9}(-1 + 4 + 3 + 4 - 1) = 1 \\
\sum(ja_j) &= \frac{1}{9}[2 - 4 + 0 + 4 - 2] = 0 \\
\sum(j^2a_j) &= \frac{1}{9}[-4 + 4 + 0 + 4 - 4] = 0 \\
\sum(j^3a_j) &= \frac{1}{9}[8 - 4 + 0 + 4 - 8] = 0
\end{aligned}
\tag{16}
$$

That means the filter passes 3-degree polynomials.

To show the filter eliminates seasonal components with period 3, $S_t = S_{t+3}$ and $\sum_{t=1}^{3}(S_t) = 0$

$$
\begin{aligned}
\hat{S}_t = \sum_{j=-2}^{2} a_j S_{t+j} =&\frac{1}{9}[-1S_{t-2} + 4S_{t-1} + 3S_t + 4S_{t+1} - S_{t+2}] \\
=&\frac{1}{9}[-S_{t+1} + 4S_{t+2} + 3S_t + 4S_{t+1} - S_{t+2}] \\
=&\frac{1}{3}\sum_{t=1}^{3}(S_t) \\
=&0
\end{aligned}
\tag{17}
$$

Problem 1.15 Solution:

Part (a)

$$\begin{aligned}
\nabla \nabla_{12} X_t =&(1-B)(1-B^{12})(a+bt+s_t+Y_t) \\
=&(1-B-B^{12}+B^{13})(a+bt+s_t+Y_t) \\
=&(a+bt+s_t+Y_t)-(a+b(t-1)+s_{t-1}+Y_{t-1}) \\
&-(a+b(t-12)+s_{t-12}+Y_{t-12})+(a+b(t-13)+s_{t-13}+Y_{t-13}) \\
=&b(t-t+1-t+12+t-13)+(s_t-s_{t-1}-s_{t-12}+s_{t-13}) \\
&+(Y_t-Y_{t-1}-Y_{t-12}+Y_{t-13}) \\
=&Y_t-Y_{t-1}-Y_{t-12}+Y_{t-13}
\end{aligned} \tag{18}$$

Therefore, we can get the mean function is:

$$E(\nabla \nabla_{12} X_t) = E(Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}) = 0 \tag{19}$$

And

$$\begin{aligned}
&Cov(\nabla \nabla_{12} X_{t+h}, \nabla \nabla_{12} X_t) \\
=&E[(Y_{t+h}-Y_{t-1+h}-Y_{t-12+h}+Y_{t-13+h})(Y_t-Y_{t-1}-Y_{t-12}+Y_{t-13})] \\
=&E[Y_{t+h}Y_t]-E[Y_{t+h}Y_{t-1}]-E[Y_{t+h}Y_{t-12}]+E[Y_{t+h}Y_{t-13}] \\
&-E[Y_{t-1+h}Y_t]+E[Y_{t-1+h}Y_{t-1}]+E[Y_{t-1+h}Y_{t-12}]-E[Y_{t-1+h}Y_{t-13}] \\
&-E[Y_{t-12+h}Y_t]+E[Y_{t-12+h}Y_{t-1}]+E[Y_{t-12+h}Y_{t-12}]-E[Y_{t+h}Y_{t-13}] \\
&+E[Y_{t-13+h}Y_t]-E[Y_{t-13+h}Y_{t-1}]-E[Y_{t-13+h}Y_{t-12}]+E[Y_{t-13+h}Y_{t-13}] \\
=&\gamma(h)-\gamma(h+1)-\gamma(h+12)+\gamma(h+13) \\
&-\gamma(h-1)+\gamma(h)+\gamma(h+11)-\gamma(h+12) \\
&-\gamma(h-12)+\gamma(h-11)+\gamma(h)-\gamma(h+1) \\
&+\gamma(h-13)-\gamma(h-12)-\gamma(h-1)+\gamma(h) \\
=&4\gamma(h)-2\gamma(h-1)-2\gamma(h+1)+\gamma(h-11)+\gamma(h+11)-2\gamma(h-12) \\
&-2\gamma(h+12)+\gamma(h-13)+\gamma(h+13)
\end{aligned} \tag{20}$$

Where $\gamma(h)$ is ACVF. From above, we can see that it is stationary time series.

Part (b)

$$\begin{aligned}
\bigtriangledown_{12}^{2}X_t =&(1-B^{12})^2((a+bt)s_t+Y_t)\\
=&(1-2B^{12}+B^{24})((a+bt)s_t+Y_t)\\
=&((a+bt)s_t+Y_t)\\
&-2((a+b(t-12))s_{t-12}+Y_{t-12})\\
&+((a+b(t-24))s_{t-24}+Y_{t-24})\\
=&(as_t-2as_t+as_t)+(bts_t-2bts_t+24bs_t+bts_t-24bs_t\\
&+(Y_t-2Y_{t-12}+Y_{t-24})\\
=&Y_t-2Y_{t-12}+Y_{t-24}
\end{aligned} \tag{21}$$

Therefore, we can get the mean function is:

$$E(\bigtriangledown_{12}^{2}X_t)=E(Y_t-2Y_{t-12}+Y_{t-24})=0 \tag{22}$$

And

$$\begin{aligned}
&Cov(\bigtriangledown_{12}^{2}X_{t+h},\bigtriangledown_{12}^{2}X_t)\\
=&E[(Y_{t+h}-2Y_{t-12+h}+Y_{t-24+h})(Y_t-2Y_{t-12}+Y_{t-24})]\\
=&E[Y_{t+h}Y_t]-2E[Y_{t+h}Y_{t-12}]+E[Y_{t+h}Y_{t-24}]\\
&-2E[Y_{t+h-12}Y_t]+4E[Y_{t-12+h}Y_{t-12}]-2E[Y_{t-12+h}Y_{t-24}]\\
&+E[Y_{t-24+h}Y_t]-2E[Y_{t-24+h}Y_{t-12}]+E[Y_{t-24+h}Y_{t-24}]\\
=&\gamma(h)-2\gamma(h+12)+\gamma(h+24)-2\gamma(h-12)\\
&+4\gamma(h)-2\gamma(h+12)+\gamma(h+24)+\gamma(h-24)\\
&-2\gamma(h-12)+\gamma(h)\\
=&6\gamma(h)-4\gamma(h+12)+\gamma(h+24)+\gamma(h-24)
\end{aligned} \tag{23}$$

Where $\gamma(h)$ is ACVF. From above, we can see that it is stationary time series.

Problem 2.1 Solution:

The best predictor is in the sense of MSE. So let the best predictor of X_{n+h} be $l(X_n)=aX_n+b$. Then it minimize $E[X_{n+h}-aX_n-b]^2$

$$\begin{cases}\frac{\partial E(X_{n+h}-aX_n-b)^2}{\partial a} & =-2E(X_{n+h}-aX_n-b)(X_n)\\ \frac{\partial E(X_{n+h}-aX_n-b)^2}{\partial b} & =-2E(X_{n+h}-aX_n-b)\end{cases}$$

Then we have:

$$\begin{cases}E(X_{n+h}X_n)-\hat{a}E(X_n^2)-\hat{b}E(X_n)=0\\ E(X_{n+h})-\hat{a}E(X_n)-\hat{b}=0\end{cases}$$

Therefore, we can get:

$$\begin{cases} \hat{a} = \frac{\rho(h)\sigma^2}{\sigma^2} = \rho(h) \\ \hat{b} = \mu(1-\hat{a}) = \mu(1-\rho(h)) \end{cases}$$

Problem 2.2 Solution:

The mean function is:

$$\begin{aligned} \mu_X(t) =& E(X_t) \\ =& E(A\cos(wt) + B\sin(wt)) \\ =& \cos(wt)E(A) + \sin(wt)E(B) \\ =& 0 \end{aligned} \tag{24}$$

And the autocovariance function is:

$$\begin{aligned} \gamma_X(h) =& Cov(X_{t+h}, X_t) \\ =& E[(X_{t+h} - E(X_{t+h}))(X_t - E(X_t))] \\ =& E(X_{t+h}X_t) \\ =& E[(A\cos(w(t+h)) + B\sin(w(t+h)))(A\cos(wt) + B\sin(wt))] \\ =& E[A^2\cos(w(t+h))\cos(wt) + AB\sin(w(2t+h)) + B^2\sin((t+h)w)\sin(wt)] \\ =& \cos(w(t+h))\cos(wt)E(A^2) + \sin(w(2t+h))E(AB) \\ &+ \sin((t+h)w)\sin(wt)E(B^2) \\ =& \cos(w(t+h))\cos(wt) + \sin(w(t+h))\sin(wt) \\ =& \cos(wh) \end{aligned} \tag{25}$$

Which is independent of t, that means X_t is stationary. And we can see that $\kappa(h)$ is autocovariance function of it, that means it is nonnegative definite.

Problem 2.3 Solution:

Part (a): The ACVF is

$$\begin{aligned}
\gamma_X(h) =& Cov(X_{t+h}, X_t) \\
=& E(X_{t+h} - EX_{t+h})(X_t - EX_t) \\
=& E(Z_{t+h} + 0.3Z_{t+h-1} - 0.4Z_{t+h-2} - 0)(Z_t + 0.3Z_{t-1} - 0.4Z_{t-2} - 0) \\
=& E(Z_{t+h} + 0.3Z_{t+h-1} - 0.4Z_{t+h-2})(Z_t + 0.3Z_{t-1} - 0.4Z_{t-2}) \\
=& E[Z_{t+h}Z_t + 0.3Z_{t+h}Z_{t-1} - 0.4Z_{t+h}Z_{t-2} \\
& + 0.3Z_{t+h-1}Z_t + 0.09Z_{t+h-1}Z_{t-1} - 0.12Z_{t+h-1}Z_{t-2} \\
& - 0.4Z_{t+h-2}Z_t - 0.12Z_{t+h-2}Z_{t-1} + 0.16Z_{t+h-2}Z_{t-2}] \\
=& \begin{cases} 1.25, & \text{if } h = 0; \\ 0.18, & \text{if } |h| = 1; \\ -0.4, & \text{if } |h| = 2; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{26}$$

Part (b): The ACVF is

$$\begin{aligned}
\gamma_X(h) =& Cov(Y_{t+h}, Y_t) \\
=& E(Y_{t+h} - EY_{t+h})(Y_t - EY_t) \\
=& E(\tilde{Z}_{t+h} - 1.2\tilde{Z}_{t+h-1} - 1.6\tilde{Z}_{t+h-2} - 0)(\tilde{Z}_t - 1.2\tilde{Z}_{t-1} - 1.6\tilde{Z}_{t-2} - 0) \\
=& E(\tilde{Z}_{t+h} - 1.2\tilde{Z}_{t+h-1} - 1.6\tilde{Z}_{t+h-2})(\tilde{Z}_t - 1.2\tilde{Z}_{t-1} - 1.6\tilde{Z}_{t-2}) \\
=& E[\tilde{Z}_{t+h}\tilde{Z}_t - 1.2\tilde{Z}_{t+h}\tilde{Z}_{t-1} - 1.6\tilde{Z}_{t+h}\tilde{Z}_{t-2} \\
& - 1.2\tilde{Z}_{t+h-1}\tilde{Z}_t + 1.44\tilde{Z}_{t+h-1}\tilde{Z}_{t-1} + 1.92\tilde{Z}_{t+h-1}\tilde{Z}_{t-2} \\
& - 1.6\tilde{Z}_{t+h-2}\tilde{Z}_t + 1.92\tilde{Z}_{t+h-2}\tilde{Z}_{t-1} + 2.56\tilde{Z}_{t+h-2}\tilde{Z}_{t-2}] \\
=& \begin{cases} 1.25, & \text{if } h = 0; \\ 0.18, & \text{if } |h| = 1; \\ -0.4, & \text{if } |h| = 2; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{27}$$

Compare with part (a), we can see that the Auto Covariance Functions are the same.

Problem 2.5 Solution:

We firstly prove convergence in mean square. Because it is a stationary time series, we suppose $E[X_t] = \mu$. By Cauchy Criterion, we can get:

$$\begin{aligned}
&\lim_{m,p \rightharpoonup \infty} E[S_m - S_{m+p}]^2 \\
=&\lim_{m,p \rightharpoonup \infty} E[\sum_{j=m+1}^{m+p} \theta^j X_{n-j}]^2 \\
=&\lim_{m,p \rightharpoonup \infty} E[\sum_{j=m+1}^{m+p} \theta^j X_{n-j}]^2 \\
=&\lim_{m,p \rightharpoonup \infty} E[\sum_{j=m+1}^{m+p} \sum_{l=m+1}^{m+p} \theta^j \theta^l X_{n-j} X_{n-l}] \\
=&\lim_{m,p \rightharpoonup \infty} \sum_{j=m+1}^{m+p} \sum_{l=m+1}^{m+p} \theta^j \theta^l E[X_{n-j} X_{n-l}] \\
=&\lim_{m,p \rightharpoonup \infty} \sum_{j=m+1}^{m+p} \sum_{l=m+1}^{m+p} \theta^j \theta^l [\gamma(j-l) + \mu^2] \\
\leq&\lim_{m,p \rightharpoonup \infty} \sum_{j=m+1}^{m+p} \sum_{l=m+1}^{m+p} |\theta^{j+l}||[\gamma(j-l) + \mu^2]| \\
\leq&\lim_{m,p \rightharpoonup \infty} \sum_{j=m+1}^{m+p} \sum_{l=m+1}^{m+p} |\theta^{j+l}|[|\gamma(j-l)| + \mu^2] \\
\leq&\lim_{m,p \rightharpoonup \infty} \sum_{j=m+1}^{m+p} \sum_{l=m+1}^{m+p} |\theta^{j+l}|[\gamma(0) + \mu^2] \\
=&0 \qquad\qquad\qquad\qquad (\text{because } |\theta| \leq 1)
\end{aligned} \tag{28}$$

Now we prove it is absolutely convergent as $m \rightharpoonup 0$. The absolutely convergence here means the series $\sum_{j=1}^{\infty} |\theta^j||X_{n-j}| < \infty$ almost surely i.e. with probability 1. It is equivalent to prove

$$E[\sum_{j=1}^{\infty} |\theta^j||X_{n-j}|] = \sum_{j=1}^{\infty} |\theta^j| E[|X_{n-j}|] < \infty \tag{29}$$

And:

$$\begin{aligned}
&\sum_{j=1}^{\infty} |\theta^j| E[|X_{n-j}|] \\
\leq &\sum_{j=1}^{\infty} |\theta^j| \sqrt{E[|X_{n-j}|]^2} \\
\leq &\sum_{j=1}^{\infty} |\theta^j| \sqrt{\gamma(0) + \mu^2} \\
\leq &\infty \qquad\qquad\qquad\qquad \text{(because } |\theta| \leq 1\text{)}
\end{aligned} \tag{30}$$

Therefore, we can have absolute convergence.

Problem 2.14 Solution:

Part (a) Solution: Now we have the stationary time series

$$X_t = A\cos(wt) + B\sin(wt)$$

with $\mu(t) = 0$ and $\gamma_X(h) = \cos(wh)$

By the property of $P_n X_{n+h}$, we know

$$\begin{aligned}
P_1 X_{1+1} = &\mu + a_1(X_1 + \mu) \\
= &\mu + \frac{\gamma_X(1)}{\gamma_X(0)}(X_1 + \mu) \\
= &\cos(w) X_1
\end{aligned} \tag{31}$$

And its mean square error is:

$$\begin{aligned}
E[X_2 - P_1 X_2]^2 = &\gamma(0) - a_1 \gamma_1(1) \\
= &1 - \cos^2 w \\
= &\sin^2 w
\end{aligned} \tag{32}$$

Part (b) Solution:

$$\begin{aligned}
P_2 X_{2+1} = &\mu + a_1(X_2 - \mu) + a_2(X_1 - \mu) \\
= &\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} \\
= &\gamma_2(1)\Gamma_2^{-1} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} \\
= &\begin{bmatrix} \gamma(1) & \gamma(2) \end{bmatrix} \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix}^{-1} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} \\
= &\begin{bmatrix} \cos(w) & \cos(2w) \end{bmatrix} \frac{1}{\sin^2(w)} \begin{bmatrix} 1 & -\cos(w) \\ -\cos(w) & 1 \end{bmatrix} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} \\
= &2\cos(w) X_2 - X_1
\end{aligned} \tag{33}$$

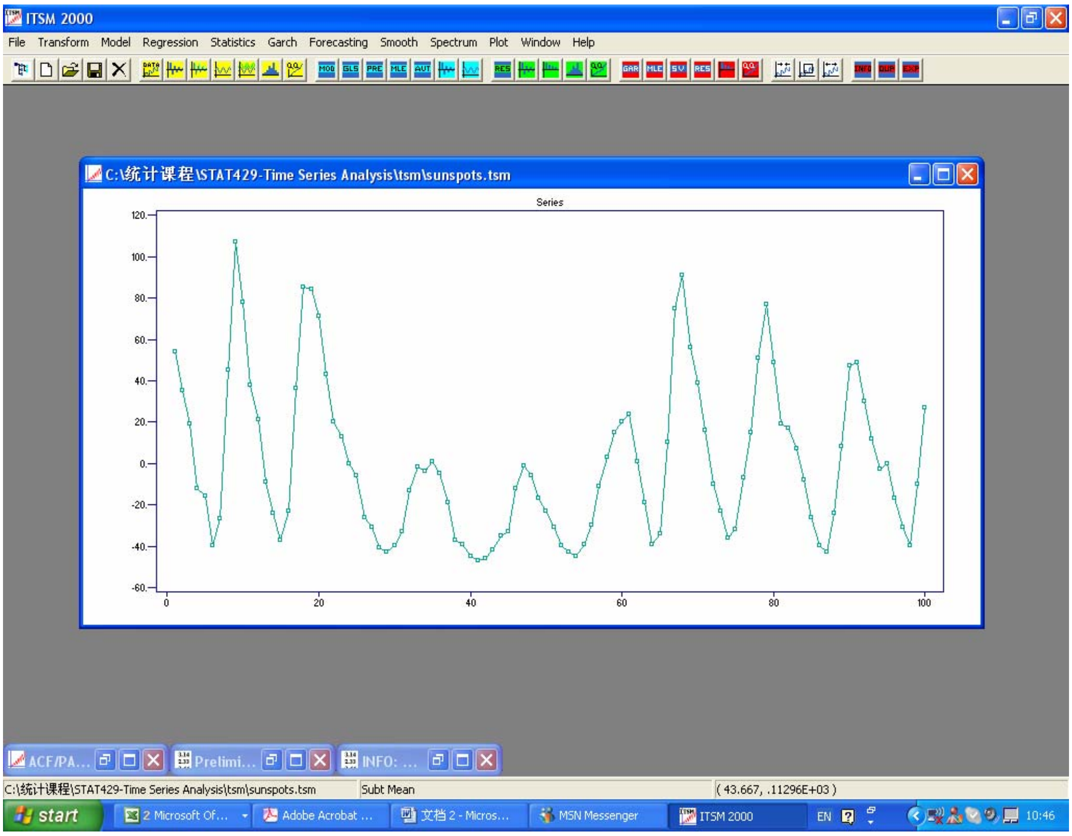
And its mean square error is:

$$
\begin{aligned}
E[X_3 - P_2 X_3]^2 =& \gamma(0) - a_2' \gamma_2(1) \\
=& 1
\end{aligned}
\tag{34}
$$

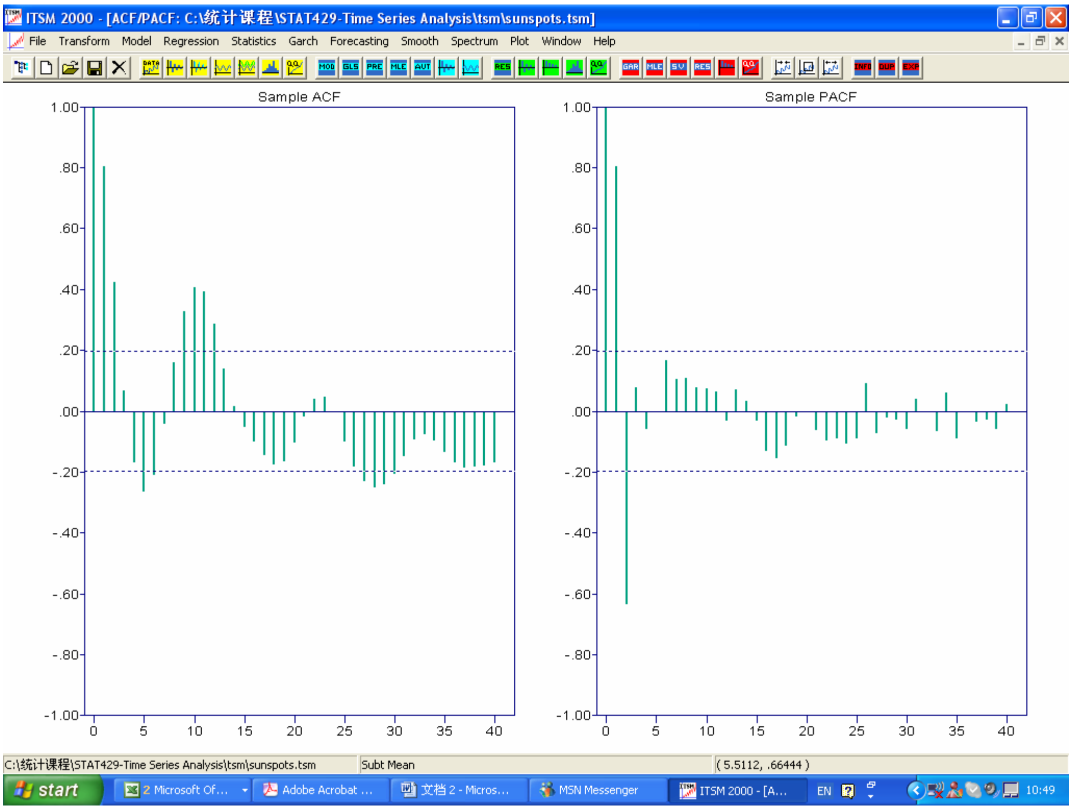
Part (c) Solution: Because we have $X_{n+1} = 2\cos(w)X_n - X_{n-1}$, therefore, by properties of $\tilde{P}_n$, we can get $\tilde{P}_n X_{n+1} = X_{n+1}$, and its mean square error is 0.

Problem 2.16:

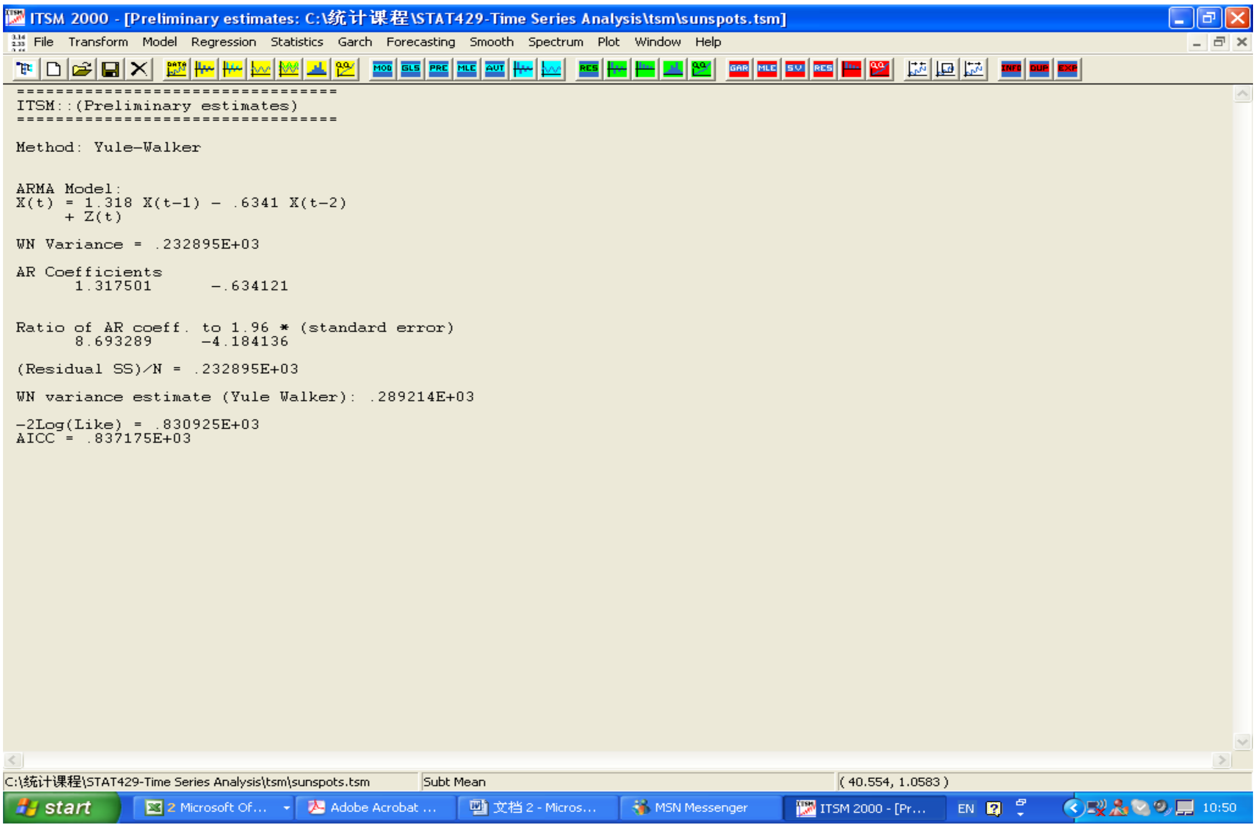
The data plot is like this:



And the sample ACF and PACF plots are:



The AR(2) model we fit is:



And finally, the Model ACF and model PACF compare with Sample ACF and PACF is: