STAT 429 Time Series Problem Set II Solution

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Problem 1 (3.4) Solution: The AR(2) Time series we have is:

$$
\begin{aligned}
&X_t = 0.8X_{t-2} + Z_t \\
\Leftrightarrow &X_t - 0.8X_{t-2} = Z_t \\
\Leftrightarrow &(1-0.8B^2)X_t = Z_t \\
\Leftrightarrow &X_t = \frac{1}{1-0.8B^2} Z_t \\
\Leftrightarrow &X_t = Z_t \sum_{j=0}^{\infty} 0.8^j B^{2j} \\
\Leftrightarrow &X_t = \sum_{j=0}^{\infty} 0.8^j Z_{t-2j}
\end{aligned}
$$

Therefore, it is causal, and the ACVF is:

$$\gamma_{X_t}(h) = Cov(X_{t+h}, X_t) = \begin{cases} \sigma^2 \frac{0.8^k}{1-0.8^2}, & \text{if } |h| = 2k; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, it is stationary, and ACF is:

$$\rho(h) = Cor(X_{t+h}, X_t) = \begin{cases} 0.8^k, & \text{if } |h| = 2k; \\ 0 & \text{otherwise.} \end{cases}$$

About the PACF, by definition we have $\alpha(0) = 1$, and $\alpha(1) = 0$. By Example 3.2.6 in the book, we have $\alpha(2) = 0.8$, and $\alpha(h) = 0$ for $h > 2$

Problem 1 (5.1) Solution

part (a) We have

$$
\begin{aligned}
\begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} &= \begin{bmatrix} 1382.2, 1114.4 \\ 1114.4, 1382.2 \end{bmatrix} \begin{bmatrix} 1114.4 \\ 591.73 \end{bmatrix} \\
&= \begin{bmatrix} 1.3175 \\ -0.6342 \end{bmatrix}
\end{aligned}
$$

and $\hat{\sigma^2} = 289.1792$

The 95 per. Asymptotic Confidence Interval for ϕ_1 is $(1.17, 1.47)$ and for ϕ_2 is $(-0.79, -0.48)$

Problem 1 (5.3) Solution

part (a). To be causal, means that

$$1 - \phi z - \phi^2 z^2 \neq 0$$

, when $|z| \leq 1$. That means

$$|\phi| < \frac{\sqrt{5} - 1}{2}$$

part (b) We have

$$\rho(1) = \rho(0)\hat{\phi} + \rho(1)\hat{\phi}^2$$

Then we can get $\hat{\phi} = 0.5087$, and

$$\hat{\sigma}^2 = \gamma(0)(1 - \hat{\rho}_2(\hat{\phi}, \hat{\phi}^2)^T) = 2.9856$$

Problem 2 Solution

(a) The time series is AR(1) Process, which is $X_t = \phi X_{t-1} + Z_t^X$. Then

$$\begin{aligned}
X_t &= \phi X_{t-1} + Z_t^X \\
\Leftrightarrow X_t - \phi X_{t-1} &= Z_t^X \\
\Leftrightarrow (1 - \phi B) X_t &= Z_t^X \\
\Leftrightarrow X_t &= \frac{1}{1 - \phi B} Z_t^X \\
\Leftrightarrow X_t &= Z_t^X \sum_{j=0}^{\infty} \phi^j B^j \\
\Leftrightarrow X_t &= \sum_{j=0}^{\infty} \phi^j Z_{t-j}^X
\end{aligned}$$

And here $|\phi| < 1$ is necessary to ensure we can do Taylor expansion.

(b) Then

$$\begin{aligned}
E(X_t) =& E(\sum_{j=0}^{\infty} \phi^j Z_{t-j}^X) \\
=& \sum_{j=0}^{\infty} \phi^j E(Z_{t-j}^X) \\
=& 0
\end{aligned}$$

And,

$$
\begin{aligned}
Var(X_t) =& Var(\sum_{j=0}^{\infty} \phi^j Z_{t-j}^X) \\
=& \sum_{j=0}^{\infty} \phi^{2j} \sigma^2 \\
=& \frac{\sigma^2}{1-\phi^2}
\end{aligned}
$$

(c) And,

$$
\begin{aligned}
Cov(X_t, X_{t+h}) =& Cov(\sum_{j=0}^{\infty} \phi^j Z_{t-j}^X, \sum_{j=0}^{\infty} \phi^j Z_{t+h-j}^X) \\
=& \sum_{j=0}^{\infty} \phi^{2j} \sigma^2 * \phi^{|h|} \\
=& \frac{\sigma^2 \phi^{|h|}}{1-\phi^2}
\end{aligned}
$$

So ACVF is:

$$
\gamma_X(h) = \frac{\sigma^2 \phi^{|h|}}{1-\phi^2}
$$

(d),(e)and (f): It is similar to part(a), we just replace the backward operator by the forward operator, then we follow the similar steps, we get the

$$
Y_t = \sum_{j=0}^{\infty} \phi^j Z_{t+j}^Y
$$

$$
E(Y_t) = 0
$$

$$
Var(Y_t) = \frac{\sigma^2}{1-\phi^2}
$$

and

$$
\gamma_Y(h) = \frac{\sigma^2 \phi^{|h|}}{1-\phi^2}
$$

Problem 3 Solution:
Part (a) So here for ϵ_t, it is a AR(1) model. therefore, we get

$$
\epsilon_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}
$$

, then we have

$$X_t = \mu + \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$

. Because of IID $N(0, \sigma^2)$ for Z_t, we can get $E(X_t) = \mu$ and,

$$\begin{aligned}
Cov(X_t, X_{t+h}) =& Cov(\sum_{j=0}^{\infty} \phi^j Z_{t-j} + \mu, \sum_{j=0}^{\infty} \phi^j Z_{t+h-j} + \mu) \\
=& \sum_{j=0}^{\infty} \phi^{2j} \sigma^2 * \phi^{|h|} \\
=& \frac{\sigma^2 \phi^{|h|}}{1 - \phi^2}
\end{aligned}$$

Then we can see that,

$$Var(X_t) = \frac{\sigma^2}{1 - \phi^2}$$

So it is stationary.

$$\begin{aligned}
Cov(\bar{X}, \bar{X}) =& \frac{1}{T^2} Cov(\sum_{t=1}^{T} X_t, \sum_{t=1}^{T} X_t) \\
=& \frac{1}{T^2} \sum_{i,j=1}^{T} Cov(X_i, X_j) \\
=& \frac{1}{T^2} \sum_{i,j=1}^{T} \gamma_X(i - j) \\
=& \frac{1}{T^2} \sum_{i,j=1}^{T} \frac{\sigma^2 \phi^{|i-j|}}{1 - \phi^2} \\
=& \frac{\sigma^2}{T^2(1 - \phi^2)} (T + \frac{2\phi}{1 - \phi}(T - \frac{1 - \phi^T}{1 - \phi}))
\end{aligned}$$

Therefore, the Asymptotic 95 percent Confidence Interval for ν is:

$$\bar{X} \pm 1.96 \sqrt{Var(\bar{X})}$$

Part (C) If we ignore the correlation in X_t, then the 95 percent CI is

$$\bar{X} \pm 1.96 \frac{\sigma}{\sqrt{T(1 - \phi^2)}}$$

So if $\phi > 0$, then the interval will tend to be smaller, and if $\phi < 0$, then the CI will expand.

Problem 4 Solution
Part (a)

$$X_t = 0.9X_{t-1} + Z_t$$

is AR(1) model, then we can have the ACF is:

$$\rho(h) = 0.9^{|h|}$$

And because $P[X_{n+1}|X_1, \cdots, X_n] = 0.9X_n$. We can have

$$\alpha(h) = \begin{cases} 1, & \text{if } h = 0 \\ 0.9 & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Part(b) $X_t = Z_t + 0.5Z_{t-1}$ which is MA(1) process,

$$\rho(h) = \begin{cases} 1, & \text{if } h = 0 \\ 0.4 & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

And PACF is:

$$\alpha(h) = \begin{cases} 1, & \text{if } h = 0 \\ 0.4 & \text{if } h = \pm 1 \\ \phi_{hh} & \text{otherwise} \end{cases}$$

Part (c) First of all, by Taylor expansion, we can have

$$X_t = \sum_{j=0}^{\infty} (\frac{8}{13}(-0.8)^j + \frac{5}{13}(0.5)^j) Z_{t-j}$$

We can have the ACF is:

$$\rho(h) = c_1(-1.25)^{|h|} + c_2 2^{|h|}$$

Where $C_1 = -0.1403846$ and $C_2 = 0.2076923$. And PACF is:

$$\alpha(h) = \begin{cases} 1, & \text{if } h = 0 \\ -0.5 & \text{if } h = \pm 1 \\ 0.4 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

Here I give the sample ACF and PACF plots, so they are slightly different from the theoretic ones.

Problem 5 Solution:

The conditional maximum likelihood estimator for ϕ is:

$$\hat{\phi} = \frac{\sum X_t X_{t+1}}{\sum X_t^2} = 0.9207$$

And:

$$\hat{\sigma}^2 = \frac{1}{3}\sum_{t=2}^{4}(X_t - 0.9207X_{t-1})^2 = 0.1035$$

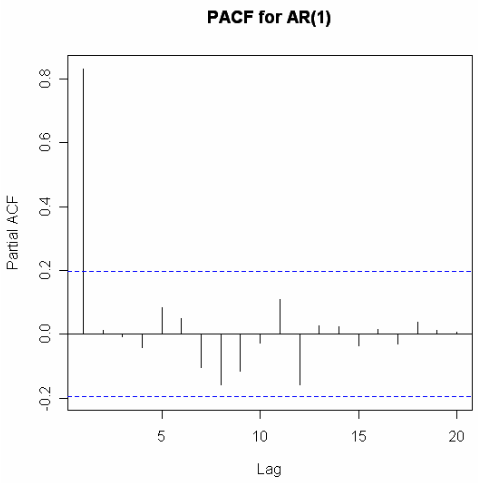
And now if we treat ϕ and $\hat{\sigma}^2$

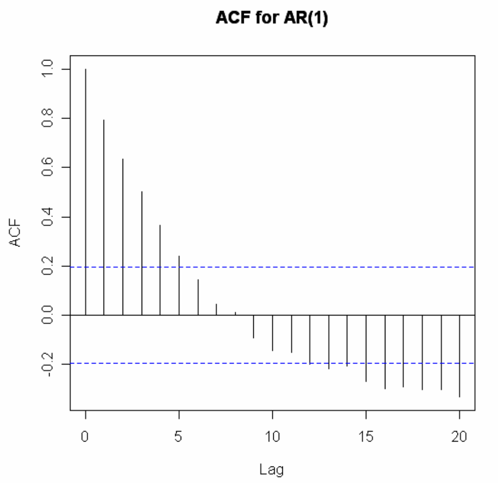
$$P[X_5|X_1, X_2, X_3, X_4] = 0.9207 * 7.8 = 7.18$$

and

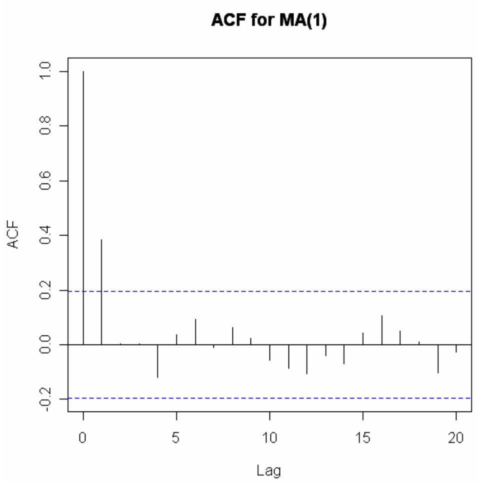
$$E([X_5] - P[X_5|X_1, X_2, X_3, X_4]) = 0$$

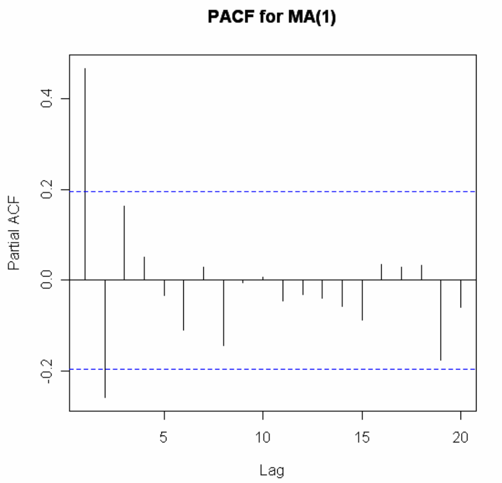
ACF AND PACF FOR PROBLEM 4 (A)





ACF AND PACF FOR PROBLEM 4 (B)





ACF AND PACF FOR PROBLEM 4 (C)