NAME:

THE UNIVERSITY OF ILLINOIS
Department of Statistics

STATISTICS 429 Fall 2005

Exam 1: Show **complete and detailed** solution to get full credit. Make sure that you have all **eight** pages of the test.

Part 1. Consider the process $\{X_t\}$ defined by $X_t = \phi_1 X_{t-1} + W_t + \theta_1 W_{t-1}$; where ϕ_1 and θ_1 are real numbers that lie in the interval $(-1, 1)$ and that $\phi_1 + \theta_1 \neq 0$; and $\{W_t\}$ is Gaussian white noise with mean 0 and variance $\sigma_W^2 = 1$.

(a) (20 points) A stationary solution of the process $\{X_t\}$ above exists and is unique. Moreover, it is causal, i.e., it can be written as $X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j}$ for some $\{\psi_j\}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$. Derive an expression for ψ_j in terms of ϕ_1 and θ_1 and show that $\sum_j |\psi_j| < \infty$.

(b) (10 points) Give $E(X_t)$ and derive an expression for $\text{Var}(X_t)$ in terms of ϕ_1 and θ_1.

(c) (10 points) Suppose that $\phi_1 = \theta_1 = 1/2$. Derive an expression for the autocovariance function $\text{Cov}(X_{t+h}, X_t)$.

(d) (5 points) Is X_t a (weakly) stationary random process? Why or why not?

Part 2. Consider a time series $Y_t = f_t + X_t$ where $f_t = \alpha_0 + \alpha_1 t$ is a non-stochastic linear trend and X_t is a weakly stationary random process whose mean is 0 and auto-covariance function is $\gamma_X(h)$. Define the first order difference time series to be $S_t = Y_t - Y_{t-1}$. We claim that S_t is weakly stationary. We prove this claim using the definition of weak stationarity.

(a) (10 points) Derive the mean of S_t.

(b) (10 points) Derive the autocovariance of S_t in terms of γ_X.

Part 3. Consider the time series Y_t defined by $Y_t = \alpha_1 X_t^{(1)} + \alpha_2 X_t^{(2)} + Z_t$ where α_1 and α_2 are non-stochastic positive real numbers in the interval $(0, 1)$ such that $\alpha_1 + \alpha_2 = 1$. Moreover, Z_t is iid $N(0, 1)$ and $X_t^{(1)}$ and $X_t^{(2)}$ are first order causal auto-regressive processes

$$\begin{aligned} X_t^{(1)} &= +0.9 X_{t-1}^{(1)} + W_t^{(1)} \\ X_t^{(2)} &= -0.9 X_{t-1}^{(2)} + W_t^{(2)} \end{aligned}$$

where $W_t^{(1)}$ and $W_t^{(2)}$ are iid Gaussian white noise with mean 0 and variance 1. We will assume that the time series $\{Z_t\}$, $\{X_t^{(1)}\}$ and $\{X_t^{(2)}\}$ are all mutually independent.

(a) (10 points) Give an expression for $\mathrm{Cov}(Y_{t+h}, Y_t)$ in terms of α_1 and α_2.

(b) (10 points). Using Y_1, find the best linear predictor for Y_2.

(c) (5 points). Give the mean squared error for the one-step predictor for Y_2.

Part 4. (5 points). Let $\{X_t\}$ be a zero mean Gaussian stationary time series with autocovariance function γ_X. Denote the data vector $\mathbf{X} = [X_2, X_3, \ldots, X_{n-1}]$ and assume that the variance-covariance matrix of $\mathbf{X}$, denoted Γ_{n-2} is invertible. Denote the best linear backcast for X_1 and the best linear forecast for X_n given $\mathbf{X}$ to be, respectively

$$\begin{aligned} P(X_1|\mathbf{X}) &= \phi^{(1)}_{n-2,1} X_2 + \ldots + \phi^{(1)}_{n-2,n-2} X_{n-1} \\ P(X_n|\mathbf{X}) &= \phi^{(2)}_{n-2,1} X_{n-1} + \ldots \phi^{(2)}_{n-2,n-2} X_2. \end{aligned}$$

Prove (or disprove) that

$$[\phi^{(1)}_{n-2,1}, \ldots, \phi^{(1)}_{n-2,n-2}] = [\phi^{(2)}_{n-2,1}, \ldots, \phi^{(2)}_{n-2,n-2}].$$

Part 5. (5 points). A problem of interest in the analysis of geophysical time series involves a simple model $X_t = S_t + AS_{t-\delta}$ where X_t is the observed time series containing a signal S_t and a reflected version of the signal with amplification factor A and time delay δ. Assume that S_t is Gaussian white noise with mean 0 and variance σ_S^2. Suppose that both A ($|A| < 1$) and δ ($\delta > 0$) are known. How can one obtain the signal S_t from the observed time series X_t?