THE UNIVERSITY OF ILLINOIS
Department of Statistics

STATISTICS 429 Fall 2006
Homework 2 - Due 17 October

1. From the book: $3.4, 5.1, 5.3$

2. Consider Models I and II that are defined to be

$$\begin{aligned} X_t &= \phi X_{t-1} + Z_t^X \\ Y_t &= \phi Y_{t+1} + Z_t^Y \end{aligned}$$

   where $|\phi| < 1$; $\{Z_t^X\}$ and $\{Z_t^Y\}$ are iid $N(0, \sigma^2)$. Model I is a "forward" model while II is a "backward" model. The goal of this exercise is to show the main result that the two models are equivalent, i.e., the mean, variance and auto-covariance structure of the two models are identical. To arrive at this conclusion, one can go through the following steps:

   (a) Show that $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}^X$.

   (b) Show that $EX_t = 0$ for all t and $\text{Var}(X_t) = \frac{\sigma^2}{1-\phi^2}$.

   (c) Show that the autocovariance of $\{X_t\}$ is $\gamma_X(h) = \phi^{|h|} \frac{\sigma^2}{1-\phi^2}$.

   (d) Show that $Y_t = \sum_{j=0}^{\infty} \phi^j Z_{t+j}^Y$.

   (e) Show that $EY_t = 0$ for all t and $\text{Var}(Y_t) = \frac{\sigma^2}{1-\phi^2}$.

   (f) Show that the autocovariance of $\{Y_t\}$ is $\gamma_Y(h) = \phi^{|h|} \frac{\sigma^2}{1-\phi^2}$.

3. Consider the model $X_t = \mu + \epsilon_t$ where μ is a constant; $\epsilon_t = \phi \epsilon_{t-1} + Z_t$; $|\phi| < 1$ and Z_t is iid $N(0, \sigma^2)$.

   (a) Show that $\{X_t\}$ is stationary. Derive its mean, variance and autocovariance.

   (b) Suppose that you are interested in estimating μ using the sample mean $\overline{X} = \frac{1}{T} \sum_{t=1}^{T} X_t$. Assume that ϕ and σ^2 are both known parameters. Derive the variance of $\overline{X}$ and form a 95% confidence interval for μ.

   (c) Discuss the implications of ignoring the correlation in X_t when computing the confidence interval. (Note than if X_t is uncorrelated over t then $\text{Var}(\overline{X}) = \frac{\sigma^2}{T}$.)

4. Sketch the auto-correlation function and partial auto-correlation function of the ff processes:

   (a) $X_t = 0.9 X_{t-1} + Z_t$

   (b) $X_t = Z_t + 0.5 Z_{t-1}$

(c) $\Phi(B)X_t = Z_t$ where $\Phi(B) = (1 + 0.8B)(1 - 0.5B)$.

5. It is believed that an AR(1) process $X_t = \phi X_{t-1} + Z_t$ where $\{Z_t\}$ is iid $N(0, \sigma^2)$ best fits a time series data $X_1 = 10, X_2 = 9.5, X_3 = 8.3, X_4 = 7.8$.

   (a) Estimate the AR(1) coefficient ϕ using the conditional maximum likelihood.

   (b) Estimate the error variance σ^2.

   (c) Treat ϕ and σ^2 as known parameters. Predict X_5 and give the estimate of the prediction error.