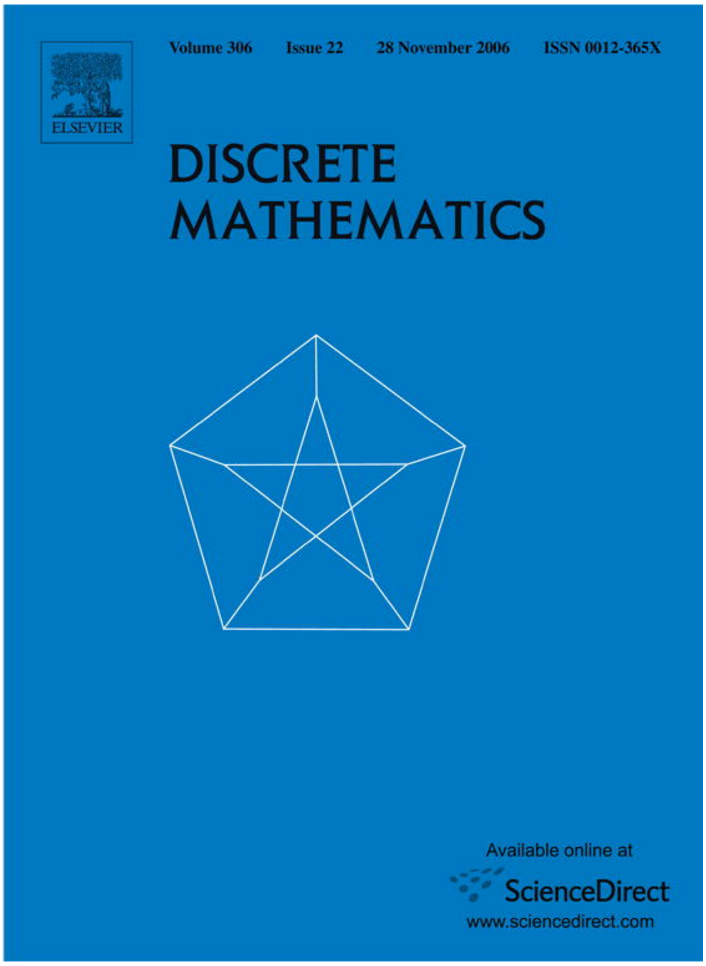
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Simple existence conditions for zero-one matrices with at most one structural zero in each row and column[☆]

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Abstract

We give simple necessary and sufficient conditions for the existence of a zero-one matrix with given row and column sums and at most one structural zero in each row and column.

Keywords: Zero-one matrix; Structural zero; Incidence matrix

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1. Introduction

Gale [5] and Ryser [7] derived necessary and sufficient conditions for the existence of an $m \times n$ zero-one matrix with given row and column sums. Fulkerson [4] considered a variation of the problem by looking at $n \times n$ square matrices with zero trace. In general, Fulkerson's conditions require checking $2^n - 1$ inequalities, which is of limited practical use. Under the strong assumption that the row and column sums are monotone together, Fulkerson simplified the conditions to n inequalities. Chen [1] further extended Fulkerson's results by showing that only n inequalities need to be checked if the row and column sums are arranged according to a certain rule.

Here we give simple necessary and sufficient conditions for the existence of an $m \times n$ zero-one matrix with given row and column sums and at most one structural zero in each row and column. Comparing to Fulkerson's [4] and Chen's [1] work, our result is more general because it applies to general $m \times n$ matrices (instead of only square matrices), not every row and column is required to have a structural zero, and the structural zeros do not have to be on the diagonal. We refer to an entry as a *structural zero* if it is constrained to be zero.

We establish the main result by ordering the rows and columns of a zero-one matrix in a special way so that a small set of inequalities are equivalent to a much larger set of inequalities. The techniques we used are more elementary compared to the methods in Fulkerson [4], who derived the conditions based on the supply-demand theorem for network flows [5], and Chen [1], who used the connection between zero-one matrices with zero trace and the graph theory.

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Zero-one matrices with structural zeros occur in many different contexts, including educational tests, ecological studies, and social networks. Many interesting statistical and mathematical problems, such as exact tests and approximating the total number of matrices, involve sampling such matrices with given marginal sums [2,8]. One efficient way to do so is to sample each matrix column by column [2,3]. The necessary and sufficient conditions for the existence of such matrices can be used to decide whether a particular configuration of a column is possible, so as to avoid generating bad matrices. The connection between zero-one matrices with zero trace and graph theory is discussed in [4].

2. Main result

Let the row sums $r_1, \ldots, r_m$ and column sums $c_1, \ldots, c_n$ be positive integers. Throughout the paper, we assume

$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j, \tag{1}$$

which is a necessary condition for $r_1, \ldots, r_m$ and $c_1, \ldots, c_n$ to be the row and column sums of a zero-one matrix. Denote the set of structural zeros by $Z = \{(x_1, y_1), \ldots, (x_s, y_s)\}$, where cell (x_i, y_i), $i = 1, \ldots, s$, is a structural zero, and there is at most one structural zero in each row and column, i.e.,

$$x_i \neq x_j, \quad y_i \neq y_j, \quad \text{if } i \neq j \text{ and } 1 \leqslant i, j \leqslant s. \tag{2}$$

We are looking for necessary and sufficient conditions for the existence of an $m \times n$ zero-one matrix $T = (t_{ij})$ such that

$$\sum_{j=1}^{n} t_{ij} = r_i, \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} t_{ij} = c_j, \quad j = 1, \ldots, n,$$

$$t_{ij} = \begin{cases} 0 & \text{if } (i, j) \in Z, \\ 0 \text{ or } 1 & \text{if } (i, j) \notin Z. \end{cases}$$

Such necessary and sufficient conditions can be deduced from the following theorem, whose proof is given in [6, p. 205].

Theorem 2.1 (*Mirsky [6]*). *Let* $0 \leqslant \rho_i' \leqslant \rho_i$, $0 \leqslant \sigma_j' \leqslant \sigma_j$, *and* $a_{ij} \geqslant 0$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, *be integers. Then there exists an* $m \times n$ *matrix* $T = (t_{ij})$ *of integral elements with row sums* $r_1, \ldots, r_m$, *and column sums* $c_1, \ldots, c_n$ *such that*

$$\begin{aligned} &\rho_i' \leqslant r_i \leqslant \rho_i, \quad 1 \leqslant i \leqslant m, \\ &\sigma_j' \leqslant c_j \leqslant \sigma_j, \quad 1 \leqslant j \leqslant n, \\ &0 \leqslant t_{ij} \leqslant a_{ij}, \quad 1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant n, \end{aligned}$$

if and only if, for all $I \subset \{1, \ldots, m\}$, $J \subset \{1, \ldots, n\}$,

$$\sum_{i \in I, j \in J} a_{ij} \geqslant \max \left\{ \sum_{i \in I} \rho_i' - \sum_{j \notin J} \sigma_j, \ \sum_{j \in J} \sigma_j' - \sum_{i \notin I} \rho_i \right\}.$$

The following corollary is a direct conclusion of Theorem 2.1.

Corollary 2.2. *The necessary and sufficient conditions for the existence of an* $m \times n$ *zero-one matrix with given row sums* $r_1, \ldots, r_m$, *column sums* $c_1, \ldots, c_n$, *and the set of structural zeros* Z, *is that for all* $I \subset \{1, \ldots, m\}$, $J \subset \{1, \ldots, n\}$,

$$\sum_{i \in I, j \in J} a_{ij} \geqslant \sum_{i \in I} r_i - \sum_{j \notin J} c_j.$$

Proof. In Theorem 2.1, let

$$\rho_i' = \rho_i = r_i, \quad i = 1, \ldots, m,$$
$$\sigma_j' = \sigma_j = c_j, \quad j = 1, \ldots, n,$$
$$a_{ij} = \begin{cases} 0 & \text{if } (i, j) \in Z, \\ 1 & \text{if } (i, j) \notin Z. \end{cases}$$

The corollary follows immediately, noting from (1) that

$$\sum_{i \in I} r_i - \sum_{j \notin J} c_j = \sum_{j \in J} c_j - \sum_{i \notin I} r_i.$$

Corollary 2.2 also applies to general configurations of Z which do not satisfy the constraints (2). In general, Corollary 2.2 involves 2^{m+n} inequalities, which makes it hard to check in practice. The next theorem simplifies the conditions to mn inequalities based on a certain ordering of the row and column sums. It is shown in Chen et al. [3] and Fulkerson [4] that ordering the row sums and column sums from largest to smallest is a useful technique for zero-one matrices. We design here a more complicated ordering scheme due to the presence of structural zeros. The goal of our construction is to put as many structural zeros as possible at the upper left corner of the matrix, while keeping the row sums and column sums in a non-increasing order.

Suppose we have an $m \times n$ zero-one matrix T with row sums $r_1, \ldots, r_m$, column sums $c_1, \ldots, c_n$, and the set of structural zeros Z. For the ith row with row sum r_i, define $y(i)$ as follows.

$$y(i) = \begin{cases} j & \text{if } (i, j) \in Z, \\ 0 & \text{if } i \notin Z_x = \{x_1, \ldots, x_s\}. \end{cases}$$

Similarly for the jth column with column sum c_j, define $x(j)$ as follows.

$$x(j) = \begin{cases} i & \text{if } (i, j) \in Z, \\ 0 & \text{if } j \notin Z_y = \{y_1, \ldots, y_s\}. \end{cases}$$

Here $y(i)$ and $x(j)$ are well defined because there is at most one structural zero in each row and column. Re-order the rows of T according to $(r_i, c_{y(i)})$, where $c_0 = 0$, i.e.,

$$\begin{aligned} &\text{the } i\text{th row should be before the } j\text{th row if} \\ &\text{(i) } r_i > r_j, \text{ or (ii) } r_i = r_j \text{ and } c_{y(i)} > c_{y(j)}. \end{aligned} \tag{3}$$

Denote the re-ordered row sums as

$$r_{u_1}, \ldots, r_{u_m}, \tag{4}$$

where $\{u_1, \ldots, u_m\}$ is a permutation of $\{1, \ldots, m\}$ that satisfies rule (3). Similarly re-order the columns according to $(c_j, r_{x(j)})$, where $r_0 = 0$, i.e.,

$$\begin{aligned} &\text{the } j\text{th column should be before the } k\text{th column if} \\ &\text{(i) } c_j > c_k, \text{ or (ii) } c_j = c_k \text{ and } r_{x(j)} > r_{x(k)}, \text{ or} \\ &\text{(iii) } c_j = c_k, \; r_{x(j)} = r_{x(k)}, \text{ and row } x(j) \text{ is before row } x(k) \text{ in (4).} \end{aligned} \tag{5}$$

Denote the re-ordered column sums as

$$c_{v_1}, \ldots, c_{v_n}, \tag{6}$$

where $\{v_1, \ldots, v_n\}$ is a permutation of $\{1, \ldots, n\}$ that satisfies rule (5). The positions of structural zeros in Z change accordingly, but there is still at most one structural zero in each row and column. Denote the new positions of structural zeros as

$$Z^* = \{(x_1^*, y_1^*), \ldots, (x_s^*, y_s^*)\}.$$

Use T^* to represent the re-ordered $m \times n$ zero-one matrix with the above row sums (4), column sums (6), and the set of structural zeros Z^*. Obviously, if there exists a zero-one matrix T^*, it can be transformed back to an $m \times n$

zero-one matrix T with row sums $r_1, \dots, r_m$, column sums $c_1, \dots, c_n$, and the set of structural zeros Z, and vice versa. Therefore, in the following we focus on the existence conditions for T^*. $\square$

Theorem 2.3. *The necessary and sufficient conditions for the existence of an $m \times n$ zero-one matrix with given row sums $r_1, \dots, r_m$, column sums $c_1, \dots, c_n$, and the set of structural zeros Z, is that*

$$\sum_{i=1}^{k}\sum_{j=1}^{l} a_{u_i v_j} \geqslant \sum_{i=1}^{k} r_{u_i} - \sum_{j=l+1}^{n} c_{v_j}, \quad k = 1, \dots, m; \quad l = 1, \dots, n, \tag{7}$$

where $r_{u_1}, \dots, r_{u_m}$ and $c_{v_1}, \dots, c_{v_n}$ are the re-ordered row and column sums defined in (4) *and* (6).

Proof. If we can show that (7) is equivalent to

$$\sum_{i\in I, j\in J} a_{u_i v_j} \geqslant \sum_{i\in I} r_{u_i} - \sum_{j\notin J} c_{v_j}, \quad I \subset \{1, \dots, m\}, \quad J \subset \{1, \dots, n\}, \tag{8}$$

then (7) is the necessary and sufficient conditions for the existence of an $m \times n$ zero-one matrix T^* with row sums (4), column sums (6), and the set of structural zeros Z^*. The theorem is thus proved following the discussion in the paragraph before Theorem 2.3.

The rest of the proof focuses on establishing the equivalence between (7) and (8). The idea is to show that for any $I \subset \{1, \dots, m\}$ of size k ($1 \leqslant k \leqslant m$) and $J \subset \{1, \dots, n\}$ of size l ($1 \leqslant l \leqslant n$),

$$\begin{aligned}&\sum_{i\in I} r_{u_i} - \sum_{j\notin J} c_{v_j} - \sum_{i\in I, j\in J} a_{u_i v_j}\\ &\quad = \sum_{i\in I} r_{u_i} + \sum_{j\in J} c_{v_j} - \sum_{j=1}^{n} c_{v_j} - (|I||J| - |(u_I, v_J) \cap Z^*|)\end{aligned} \tag{9}$$

is maximized when $I = \{1, \dots, k\}$ and $J = \{1, \dots, l\}$, where

$$(u_I, v_J) = \{(u_i, v_j) : \ i \in I, \ \ j \in J\}.$$

This is equivalent to showing that

$$\begin{aligned}&\sum_{i=1}^{k} r_{u_i} + \sum_{j=1}^{l} c_{v_j} + |(u_{\{1,\dots,k\}}, v_{\{1,\dots,l\}}) \cap Z^*|\\ &\quad \geqslant \sum_{i\in I} r_{u_i} + \sum_{j\in J} c_{v_j} + |(u_I, v_J) \cap Z^*|\end{aligned} \tag{10}$$

for any $I \subset \{1, \dots, m\}$ and $J \subset \{1, \dots, n\}$. Let $d = |(u_I, v_J) \cap Z^*|$. Inequality (10) is obviously true when $d \leqslant |(u_{\{1,\dots,k\}}, v_{\{1,\dots,l\}}) \cap Z^*|$ because

$$r_{u_1} \geqslant \cdots \geqslant r_{u_m}, \quad c_{v_1} \geqslant \cdots \geqslant c_{v_n}. \tag{11}$$

In the following, assume $d > |(u_{\{1,\dots,k\}}, v_{\{1,\dots,l\}}) \cap Z^*|$. Let $I = \{p_1, \dots, p_k\} \subset \{1, \dots, m\}$ and $J = \{q_1, \dots, q_l\} \subset \{1, \dots, n\}$. Without loss of generality, assume $\{(u_{p_1}, v_{q_1}), \dots, (u_{p_d}, v_{q_d})\}$ are the d structural zeros in (u_I, v_J). We will construct permutations of $(1, \dots, k)$ and $(1, \dots, l)$ so that for each structural zero (u_{p_i}, v_{q_i}), we can find a corresponding structural zero in $(u_{\{1,\dots,k\}}, v_{\{1,\dots,l\}}) \cap Z^*$ if needed. This is crucial for proving (10).

Let $(a_1, \dots, a_k)$ be a permutation of $(1, \dots, k)$ and $(b_1, \dots, b_l)$ be a permutation of $(1, \dots, l)$, which are constructed according to the following three steps.

Step 1: Let $a_i = p_i$, if $p_i \in \{1, \dots, k\}$, $i = 1, \dots, k$. Let $b_j = q_j$, if $q_j \in \{1, \dots, l\}$, $j = 1, \dots, l$.

Step 2: For each $1 \leqslant i \leqslant d$, if $p_i > k$ or $q_i > l$, we let $a_i = g$ and $b_i = h$ if there exists (g, h) such that

$$1 \leqslant g \leqslant \min\{k, p_i\}, \quad g \notin I; \quad 1 \leqslant h \leqslant \min\{l, q_i\}, \quad h \notin J; \quad (u_g, v_h) \in Z^*, \tag{12}$$

and (g, h) has not been assigned to any (a_j, b_j) yet, $1 \leqslant j \leqslant d$. If there is more than one pair of (g, h) satisfying the above conditions, we randomly choose one and assign its value to (a_i, b_i). If there is no (g, h) satisfying the above conditions, skip this step.

Step 3: Randomly assign the available values in $\{1, \ldots, k\}$ and $\{1, \ldots, l\}$ to a_i and b_j, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$, if they are not assigned values yet, to make $(a_1, \ldots, a_k)$ a permutation of $(1, \ldots, k)$ and $(b_1, \ldots, b_l)$ a permutation of $(1, \ldots, l)$,

For each (p_i, q_i), $1 \leqslant i \leqslant d$, there are four different cases.

Case I: $1 \leqslant p_i \leqslant k$ and $1 \leqslant q_i \leqslant l$. In this case, $a_i = p_i$ and $b_i = q_i$ according to Step 1. $(u_{a_i}, v_{b_i}) \in Z^*$ because (u_{p_i}, v_{q_i}) is a structural zero. Therefore

$$r_{u_{a_i}} + c_{v_{b_i}} + |(u_{a_i}, v_{b_i}) \cap Z^*| = r_{u_{p_i}} + c_{v_{q_i}} + 1. \tag{13}$$

Case II: $1 \leqslant p_i \leqslant k$ and $q_i > l$. In this case, either (a_i, b_i) is assigned in Step 2 and satisfies conditions (12), or $a_i = p_i$ and b_i is randomly assigned in Step 3. If (a_i, b_i) is assigned in Step 2 and satisfies conditions (12), then (11) and (12) together immediately lead to

$$r_{u_{a_i}} + c_{v_{b_i}} + |(u_{a_i}, v_{b_i}) \cap Z^*| \geqslant r_{u_{p_i}} + c_{v_{q_i}} + 1. \tag{14}$$

If $a_i = p_i$ and b_i is randomly assigned in Step 3, we need to compare $c_{v_{b_i}}$ with $c_{v_{q_i}}$. If $c_{v_{b_i}} > c_{v_{q_i}}$, we have

$$r_{u_{a_i}} + c_{v_{b_i}} \geqslant r_{u_{p_i}} + c_{v_{q_i}} + 1. \tag{15}$$

If $c_{v_{b_i}} = c_{v_{q_i}}$, then based on rule (5) and the fact that $b_i \leqslant l < q_i$, there exists $1 \leqslant g(b_i) < p_i$ such that $(u_{g(b_i)}, v_{b_i})$ is a structural zero. Now $(g(b_i), b_i)$ satisfies all conditions in (12) except the requirement that $g(b_i) \notin I$. Since $a_i = p_i$ and b_i is assigned in Step 3, not Step 2, we must have $g(b_i) \in I$. Note that $b_i \notin J$ (otherwise b_i would be assigned in Step 1, instead of Step 3), so $g(b_i) = p_{e(b_i)}$ for some $d < e(b_i) \leqslant k$. Therefore

$$r_{u_{a_i}} + c_{v_{b_i}} + r_{u_{g(b_i)}} + |(u_{g(b_i)}, v_{b_i}) \cap Z^*| = r_{u_{p_i}} + c_{v_{q_i}} + r_{u_{p_{e(b_i)}}} + 1. \tag{16}$$

Case III: $p_i > k$ and $1 \leqslant q_i \leqslant l$. Using a similar argument as that for Case II, we have the corresponding versions of (14)–(16).

Case IV: $p_i > k$ and $q_i > l$. In this case, $r_{u_{a_i}} \geqslant r_{u_{p_i}}$ and $c_{v_{b_i}} \geqslant c_{v_{q_i}}$ because $a_i < p_i$ and $b_i < q_i$. If at least one of the following three conditions hold: (i) $r_{u_{a_i}} > r_{u_{p_i}}$, or (ii) $c_{v_{b_i}} > c_{v_{q_i}}$, or (iii) (u_{a_i}, v_{b_i}) is a structural zero, then we have

$$r_{u_{a_i}} + c_{v_{b_i}} + |(u_{a_i}, v_{b_i}) \cap Z^*| \geqslant r_{u_{p_i}} + c_{v_{q_i}} + 1. \tag{17}$$

If $r_{u_{a_i}} = r_{u_{p_i}}$, $c_{v_{b_i}} = c_{v_{q_i}}$, and (u_{a_i}, v_{b_i}) is not a structural zero, then based on rule (5) and the fact that $a_i < p_i$ and $b_i < q_i$, there exists $1 \leqslant g(b_i) \leqslant m$ and $1 \leqslant h(a_i) \leqslant n$ such that $(u_{g(b_i)}, v_{b_i})$ and $(u_{a_i}, v_{h(a_i)})$ are structural zeros. In the following, we show that at least one of the two inequalities is true: $g(b_i) \leqslant k$ and $h(a_i) \leqslant l$.

We will prove by contradiction. Suppose $g(b_i) > k$ and $h(a_i) > l$. Based on rule (3) and the fact that $a_i \leqslant k < p_i$ and $r_{u_{a_i}} = r_{u_{p_i}}$, we have $c_{v_{h(a_i)}} \geqslant c_{v_{q_i}}$. Since $c_{v_{b_i}} = c_{v_{q_i}}$, we immediately have

$$c_{v_{h(a_i)}} \geqslant c_{v_{b_i}}. \tag{18}$$

Based on rule (3) and the fact that $a_i \leqslant k < g(b_i)$, we have

$$r_{u_{a_i}} \geqslant r_{u_{g(b_i)}}. \tag{19}$$

Combining (18), (19), and the fact that row u_{a_i} is before row $u_{g(b_i)}$, we have that column $v_{h(a_i)}$ must be before column v_{b_i}, i.e., $h(a_i) < b_i$, based on rule (5). This contradicts the assumption that $h(a_i) > l \geqslant b_i$. Therefore $g(b_i) > k$ and $h(a_i) > l$ cannot be true simultaneously.

Without loss of generality, assume $g(b_i) \leqslant k$. Now $(g(b_i), b_i)$ satisfies all conditions in (12) except the requirement that $g(b_i) \notin I$. Since a_i and b_i are assigned in Step 3 (if they are assigned in Step 2, $(u_{a_i}, v_{b_i}) \in Z^*$), we must have $g(b_i) \in I$. Note that $b_i \notin J$ (otherwise b_i would be assigned in Step 1, instead of Step 3), so $g(b_i) = p_{e(b_i)}$ for some $d < e(b_i) \leqslant k$. Therefore

$$r_{u_{a_i}} + c_{v_{b_i}} + r_{u_{g(b_i)}} + |(u_{g(b_i)}, v_{b_i}) \cap Z^*| = r_{u_{p_i}} + c_{v_{q_i}} + r_{u_{p_{e(b_i)}}} + 1. \tag{20}$$

In every equality and inequality in the above four cases, there is an additional 1 on the right-hand side, which is crucial for proving (10) because (p_i, q_i), $1 \leqslant i \leqslant d$, is a structural zero. For $d < i \leqslant k$ and $d < j \leqslant l$, we always have

$$r_{u_{a_i}} \geqslant r_{u_{p_i}}, \quad c_{v_{b_j}} \geqslant c_{v_{q_j}}, \tag{21}$$

because of (11) and

$$\begin{aligned} &a_i = p_i \quad \text{if } p_i \leqslant k, \\ &a_i < p_i \quad \text{if } p_i > k, \\ &b_i = q_i \quad \text{if } q_i \leqslant l, \\ &b_i < q_i \quad \text{if } q_i > l. \end{aligned}$$

Combining all of above four cases together with (21), we have

$$\begin{aligned} &\sum_{i=1}^{k} r_{u_i} + \sum_{j=1}^{l} c_{v_j} + |(u_{\{1,\ldots,k\}}, v_{\{1,\ldots,l\}}) \cap Z^*| \\ &\quad = \sum_{i=1}^{k} r_{u_{a_i}} + \sum_{j=1}^{l} c_{v_{b_j}} + |(u_{\{a_1,\ldots,a_k\}}, v_{\{b_1,\ldots,b_l\}}) \cap Z^*| \\ &\quad \geqslant \sum_{i=1}^{k} r_{u_{p_i}} + \sum_{j=1}^{l} c_{v_{q_j}} + d \\ &\quad = \sum_{i \in I} r_{u_i} + \sum_{j \in J} c_{v_j} + |(u_I, v_J) \cap Z^*|, \end{aligned}$$

indicating that (10) is true. The theorem is therefore proved. □

In the above theorem, the rows and columns are re-ordered according to rules (3) and (5). The following corollary shows that it is enough to re-order the rows and columns so that both row and column sums are decreasing.

Corollary 2.4. *The necessary and sufficient conditions for the existence of an* $m \times n$ *zero-one matrix with given row sums* $r_1, \ldots, r_m$, *column sums* $c_1, \ldots, c_n$, *and the set of structural zeros* Z, *is that*

$$\sum_{i=1}^{k} \sum_{j=1}^{l} a_{u_i^* v_j^*} \geqslant \sum_{i=1}^{k} r_{u_i^*} - \sum_{j=l+1}^{n} c_{v_j^*}, \quad k = 1, \ldots, m; \quad l = 1, \ldots, n, \tag{22}$$

where $\{u_1^*, \ldots, u_m^*\}$ *is a permutation of* $\{1, \ldots, m\}$ *and* $\{v_1^*, \ldots, v_m^*\}$ *is a permutation of* $\{1, \ldots, n\}$ *such that* $r_{u_1^*} \geqslant \cdots \geqslant r_{u_m^*}$ *and* $c_{v_1^*} \geqslant \cdots \geqslant c_{v_n^*}$.

Proof. To establish the equivalence between (22) and (7), it is enough to show that for every set of neighboring rows $(r_{u_k^*}, r_{u_{k+1}^*})$ with $r_{u_k^*} = r_{u_{k+1}^*}$, we can switch these two rows while maintaining property (22), and for every set of neighboring columns $(c_{v_l^*}, c_{v_{l+1}^*})$ with $c_{v_l^*} = c_{v_{l+1}^*}$, we can switch these two columns while maintaining property (22). The reason is that after finite number of such switches, we can change the current ordering of rows $r_{u_1^*}, \ldots, r_{u_m^*}$ and columns $c_{v_1^*}, \ldots, c_{v_n^*}$ to the ordering required by Theorem 2.3, i.e., the ordering satisfying rules (3) and (5).

We prove in the following that we can switch neighboring rows with equal row sums. The proof for columns is similar. Suppose $(r_{u_k^*}, r_{u_{k+1}^*})$ are two neighboring rows with $r_{u_k^*} = r_{u_{k+1}^*}$, then (22) implies that

$$\sum_{i=1}^{k-1} \sum_{j=1}^{l} a_{u_i^* v_j^*} \geqslant \sum_{i=1}^{k-1} r_{u_i^*} - \sum_{j=l+1}^{n} c_{v_j^*}, \quad l = 1, \ldots, n, \tag{23}$$

$$\sum_{i=1}^{k-1}\sum_{j=1}^{l} a_{u_i^* v_j^*} + \sum_{j=1}^{l} a_{u_k^* v_j^*} \geqslant \sum_{i=1}^{k-1} r_{u_i^*} + r_{u_k^*} - \sum_{j=l+1}^{n} c_{v_j^*}, \quad l=1,\ldots,n, \tag{24}$$

$$\sum_{i=1}^{k+1}\sum_{j=1}^{l} a_{u_i^* v_j^*} \geqslant \sum_{i=1}^{k+1} r_{u_i^*} - \sum_{j=l+1}^{n} c_{v_j^*}, \quad l=1,\ldots,n. \tag{25}$$

To show that we can switch the two rows $(r_{u_k^*}, r_{u_{k+1}^*})$ while maintaining property (22), it is enough to prove that

$$\sum_{i=1}^{k-1}\sum_{j=1}^{l} a_{u_i^* v_j^*} + \sum_{j=1}^{l} a_{u_{k+1}^* v_j^*} \geqslant \sum_{i=1}^{k-1} r_{u_i^*} + r_{u_{k+1}^*} - \sum_{j=l+1}^{n} c_{v_j^*}, \quad l=1,\ldots,n. \tag{26}$$

We will prove (26) by contradiction. Suppose (26) does not hold, i.e.,

$$\sum_{i=1}^{k-1}\sum_{j=1}^{l} a_{u_i^* v_j^*} + \sum_{j=1}^{l} a_{u_{k+1}^* v_j^*} < \sum_{i=1}^{k-1} r_{u_i^*} + r_{u_{k+1}^*} - \sum_{j=l+1}^{n} c_{v_j^*} \quad \text{for some } 1 \leqslant l \leqslant n. \tag{27}$$

Then (27), (24) and the fact that $r_{u_{k+1}^*} = r_{u_k^*}$ lead to

$$\sum_{j=1}^{l} a_{u_{k+1}^* v_j^*} < \sum_{j=1}^{l} a_{u_k^* v_j^*}. \tag{28}$$

Note that there is at most one structural zero in each row and column, so (28) implies that

$$\sum_{j=1}^{l} a_{u_{k+1}^* v_j^*} = \sum_{j=1}^{l} a_{u_k^* v_j^*} - 1. \tag{29}$$

Combining (29), (27) and (24), we have

$$\sum_{i=1}^{k-1}\sum_{j=1}^{l} a_{u_i^* v_j^*} + \sum_{j=1}^{l} a_{u_k^* v_j^*} = \sum_{i=1}^{k-1} r_{u_i^*} + r_{u_k^*} - \sum_{j=l+1}^{n} c_{v_j^*} \quad \text{for some } 1 \leqslant l \leqslant n. \tag{30}$$

Therefore

$$\sum_{j=1}^{l} a_{u_k^* v_j^*} \leqslant r_{u_k^*}, \tag{31}$$

based on (30) and (23). Combining (31) and (27), we have

$$\begin{aligned}\sum_{i=1}^{k+1}\sum_{j=1}^{l} a_{u_i^* v_j^*} &= \sum_{i=1}^{k-1}\sum_{j=1}^{l} a_{u_i^* v_j^*} + \sum_{j=1}^{l} a_{u_{k+1}^* v_j^*} + \sum_{j=1}^{l} a_{u_k^* v_j^*}\\ &< \sum_{i=1}^{k-1} r_{u_i^*} + r_{u_{k+1}^*} - \sum_{j=l+1}^{n} c_{v_j^*} + r_{u_k^*}\\ &= \sum_{i=1}^{k+1} r_{u_i^*} - \sum_{j=l+1}^{n} c_{v_j^*},\end{aligned}$$

which contradicts (25). The corollary is thus proved.

A square matrix with zero trace corresponds to $m=n$ and a special configuration of structural zeros

$$Z = \{(1,1),(2,2),\ldots,(n,n)\},$$

which satisfies the requirement that there is at most one structural zero in each row and column. Thus the following corollary is a direct conclusion of Corollary 2.4. □

Corollary 2.5. *The necessary and sufficient conditions for the existence of an* $n \times n$ *zero-one matrix with given row sums* $r_1, \ldots, r_n$, *column sums* $c_1, \ldots, c_n$, *and zero trace, is that*

$$\sum_{i=1}^{k}\sum_{j=1}^{l} a_{u_i^* v_j^*} \geqslant \sum_{i=1}^{k} r_{u_i^*} - \sum_{j=l+1}^{n} c_{v_j^*}, \quad k = 1, \ldots, n; \quad l = 1, \ldots, n, \tag{32}$$

where $r_{u_1^*}, \ldots, r_{u_m^*}$ *and* $c_{v_1^*}, \ldots, c_{v_n^*}$ *are the re-ordered row and column sums so that* $r_{u_1^*} \geqslant \cdots \geqslant r_{u_m^*}$ *and* $c_{v_1^*} \geqslant \cdots \geqslant c_{v_n^*}$.

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